Non-critical string pentagon equations and their solutions

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We derive pentagon type relations for the 3-point boundary tachyon correlation functions in the non-critical open string theory with generic $c_{\text{matter}} < 1$ and study their solutions in the case of FZZ branes. A new general formula for the Liouville 3-point factor is derived.

1. Introduction

The associativity of the operator product expansion (OPE) of the boundary fields implies an equation [1] for the boundary 3-point functions. It can be rewritten [2] as a pentagon type relation for the boundary OPE coefficients, similar to the pentagon relation for the fusing matrix, the quantum 6j-symbols. The two equations are identified in the rational case [3] as part of the Big Pentagon relations of a weak Hopf algebra [4], [5], interpreted as the quantum symmetry of the given BCFT. The boundary field OPE coefficients play the role of the quantum 3j-symbols of this algebra. In the CFT described by diagonal modular invariants the two pentagon relations admit an identical form and thus the quantum 3j- and 6j symbols coincide up to a gauge [2], [6], confirming an earlier result in [7], where the 3-point boundary functions were computed explicitly in the $sl(2)$ case. The generalisation of the quantum 6j symbols to the non-compact Liouville theory was found in [8], and the boundary OPE coefficients with boundaries of FZZ type [9] were described in [10]. The gauge choice is correlated with the Lagrangean formulation of the theory, namely, the expression computed with the boundary $c > 25$ Coulomb gas technique of [9] is recovered as a residue from the integral formula in [10]. The pentagon equations in this case were further discussed and used in [11].

In this paper[1] we consider the non-critical string analog of the boundary pentagon relations and their solutions. The theory combines two Virasoro theories, $c < 1$ (matter) and $c > 25$ (Liouville), so that the overall central charge is compensated by the central charge of a pair of free ghost fields. As in the bulk [12], [13], the emphasis is on the presence of non-trivial matter interaction implemented conventionally by the two $c < 1$ screening charges. Our derivation here exploits only the factorisation of the 3-point tachyon boundary correlators into matter and Liouville factors. It yields the equations which can be obtained alternatively in the ground ring approach [14], [15], [16], [17], [12], using the coefficients in the OPE of the ground ring generators and the tachyons. The result is a generalisation of the trivial matter case considered in [11], in which the tachyon correlators are described by the correlators in the pure Liouville theory but with additional constraints on the set of representations arising from the mass-shell condition.

The solution of the general equations is a product of the matter and Liouville 3-point boundary coefficients. We consider the case when the matter fields are restricted by a charge conservation condition with two types of screening charges, or/and correspond to degenerate $c < 1$ Virasoro representations. In this case the matter factor is given by the Coulomb gas expression and in the non-rational case can be recovered by analytic

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continuation of the $c > 25$ Liouville Coulomb gas expression. The Liouville factor of the tachyon 3-point correlator is given in principle by the integral Ponsot-Teschner (PT) formula. We derive instead a simpler compact expression using recursively the Liouville pentagon equations. It is valid for charges corresponding to degenerate matter representations and is expressed in terms of finite sum basic hypergeometric functions of type ${}_4\Phi_3$ with bounds generically consistent with the matter fusion rules. The formula generalises a special (thermal) case result of [18]. Unlike the expression found in [18] our formula is explicitly invariant under cyclic permutations of the boundary fields.

Furthermore in Appendix B we write down the equations for the 3-point tachyon boundary correlators in the nonstandard variant of the Liouville gravity introduced in [12], [19]. These equations are obtained exploiting compositions of the general boundary ground ring OPE relations derived in [12].

2. Pentagon equations

We shall keep only the Liouville field labels for the tachyon boundary operator $T_\beta^{(\epsilon)} = {}^{(\bar\sigma_2,\sigma_2)}T_{(e,\beta)}{}^{(\bar\sigma_1,\sigma_1)}$ of chirality $\epsilon = \pm 1$, $(e = \epsilon\beta - \epsilon b^\epsilon, \beta)$, while the matter representation label e and matter boundary labels $\bar\sigma_i$ will be suppressed. The parameter b determining the central charges $c = 13 + 6(b^2 + 1/b^2) > 25$ and $c = 13 - 6(b^2 + 1/b^2) < 1$ of the two Virasoro theories is generically an arbitrary real number; most of the formulae below remain true for the rational (minimal matter) case. It is convenient to use the "leg factor" normalisation

$$T_\beta^{(\epsilon)}(x) = \Gamma(b^\epsilon(Q - 2\beta))\ \mathbf{c}(x)\, e^{2i\epsilon(\beta - b^\epsilon)\chi(x)}\ e^{2\beta\phi(x)}\,. \tag{2.1}$$

The scaling dimensions are given respectively by

$$\begin{aligned}
&\triangle_L(\beta) = \beta(Q-\beta)\,, \quad Q = 1/b + b\,,\\
&\triangle_M(e) = e(e-e_0)\,, \quad e_0 = 1/b - b\,,\\
&\triangle_M(e) + \triangle_L(\epsilon e + b^\epsilon) = 1 = -\triangle_{\text{ghost}\,\mathbf{c}}\,.
\end{aligned} \tag{2.2}$$

• The pentagon relations take simple recursive form when one of the operators corresponds to a fundamental degenerate Virasoro representation. Starting with the Liouville case, one has (see e.g., [11])

$$\begin{aligned}
&C^L_{\sigma_3\,,\beta_2 - t\frac{b}{2}}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \sigma_4 & \sigma_2\end{bmatrix} C^L_{\sigma_2=\sigma_3\pm\frac{b}{2}\,,\beta_3}\begin{bmatrix}\beta_2 - t\frac{b}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix} = \\
&F^L_{+t}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \beta_3 & \beta_1\end{bmatrix} C^L_{\sigma_3\pm\frac{b}{2}\,\beta_1 - \frac{b}{2}}\begin{bmatrix}-\frac{b}{2} & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} C^L_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1 - \frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix} + \\
&F^L_{-t}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \beta_3 & \beta_1\end{bmatrix} C^L_{\sigma_3\pm\frac{b}{2}\,\beta_1 + \frac{b}{2}}\begin{bmatrix}-\frac{b}{2} & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} C^L_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1 + \frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}\,, \qquad t = \pm 1\,.
\end{aligned} \tag{2.3}$$

The relation of the OPE coefficients to the (cyclically symmetric) 3-point correlators is

$$C^L_{\sigma_2,Q-\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} = \langle {}^{\sigma_1}B_{\beta_3}{}^{\sigma_3}\, B_{\beta_2}{}^{\sigma_2} B_{\beta_1}{}^{\sigma_1}\rangle = {}^L C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$$
$$= S(\sigma_1,\beta_3,\sigma_3)\, {}^L C^{\sigma_3,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_2,\beta_1} \tag{2.4}$$

where $S(\sigma_1,\beta_3,\sigma_3)$ is the reflection amplitude [9]. The Coulomb gas constants computed for labels $\{\beta_i\}$ restricted by the charge conservation condition $\sum_i \beta_i - Q = -mb - \frac{n}{b}$ (or any reflection of this condition) are recovered as residues from the expression in [10].

Similarly the matter pentagon equation reads

$$C^M_{\bar\sigma_3,e_2+t'\frac{b}{2}}\begin{bmatrix}e_2 & \frac{b}{2}\\ \bar\sigma_4 & \bar\sigma_2\end{bmatrix} C^M_{\bar\sigma_2=\bar\sigma_3\mp\frac{b}{2},e_3}\begin{bmatrix}e_2+t'\frac{b}{2} & e_1\\ \bar\sigma_4 & \bar\sigma_1\end{bmatrix} =$$
$$F^M_{+,-t'}\begin{bmatrix}e_2 & \frac{b}{2}\\ e_3 & e_1\end{bmatrix} C^M_{\bar\sigma_3\mp\frac{b}{2}\,e_1-\frac{b}{2}}\begin{bmatrix}\frac{b}{2} & e_1\\ \bar\sigma_3 & \bar\sigma_1\end{bmatrix} C^M_{\bar\sigma_3,e_3}\begin{bmatrix}e_2 & e_1-\frac{b}{2}\\ \bar\sigma_4 & \bar\sigma_1\end{bmatrix} + \tag{2.5}$$
$$F^M_{-,-t'}\begin{bmatrix}e_2 & \frac{b}{2}\\ e_3 & e_1\end{bmatrix} C^M_{\bar\sigma_3\mp\frac{b}{2}\,e_1+\frac{b}{2}}\begin{bmatrix}\frac{b}{2} & e_1\\ \bar\sigma_3 & \bar\sigma_1\end{bmatrix} C^M_{\bar\sigma_3,e_3}\begin{bmatrix}e_2 & e_1+\frac{b}{2}\\ \bar\sigma_4 & \bar\sigma_1\end{bmatrix}, \qquad t'=\pm 1\,,$$

and the matter constants will be normalised to be 1 for $e_1+e_2+(e_0-e_3)-e_0=0$.

- The fusion matrix elements and the boundary OPE constants in (2.3), (2.5) containing a fundamental Virasoro representation are known constants, which are recalled in Appendix A. The matter and Liouville fusion matrix elements are related by analytic continuation. E.g. for the choice of the chiralities of the three fields as $(+,-,+)$

$$\beta_3 = e_3 + b\,,\ \beta_2 = -e_2 + 1/b\,,\ \beta_1 = e_1 + b \tag{2.6}$$

one has $F^M_{s,t} = F^L_{-s,t}\,,\quad \tilde F^M_{s,t} = \tilde F^L_{-s,t}\,,$ which implies the following identities

$$F^L_{++}F^M_{--} - F^L_{+-}F^M_{-+} = 0 = F^L_{-+}F^M_{+-} - F^L_{--}F^M_{++}\,,$$
$$-F^L_{+,+}\,F^M_{+,-} + F^L_{+,-}\,F^M_{+,+} = \frac{Q-2\beta_1}{Q-2\beta_2} = -F^L_{-,-}\,F^M_{-,+} + F^L_{-,+}\,F^M_{-,-}\,. \tag{2.7}$$

Now we multiply the matter and Liouville pentagon identities (2.3) and (2.5) for the same fixed $t=t'$ - consistent with a tachyon of negative chirality $(e_2+t\frac{b}{2},\beta_2-t\frac{b}{2})$ in the l.h.s. On the other hand in the r.h.s. we get besides the two tachyon contributions also two mixed terms, inconsistent with the mass-shell condition. Due to the first of the identities in (2.7) these mixed terms are cancelled in the linear combination of the $t=+1$ and $t=-1$ product identities taken with relative minus sign. To compute this linear

combination one has to take into account the second identity (2.7) and one finally obtains for the normalised as in (2.1) tachyon OPE constants $\hat{C}$

$$\hat{C}_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2-\frac{b}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}+$$
$$\sqrt{\lambda_{_L}\,\lambda_{_M}}\;c(\beta_2)\;c^M_{(-\bar\delta)}(\bar\sigma_2=\bar\sigma_3-\bar\delta\frac{b}{2},e_2,\bar\sigma_4)\;c^L_{(\delta)}(\sigma_2=\sigma_3+\delta\frac{b}{2},\beta_2,\sigma_4)\,\hat{C}_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2+\frac{b}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}=$$

$$-\sqrt{\lambda_{_M}}\,c^M_{(\bar\delta)}(\bar\sigma_3,e_1,\bar\sigma_1)\,\hat{C}_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1-\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}-\sqrt{\lambda_{_L}}\,c^L_{(-\delta)}(\sigma_3,\beta_1,\sigma_1)\,\hat{C}_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix},\tag{2.8}$$

where $\delta\,,\bar\delta=\pm1\,,$

$$c^L_{(\mp)}(\sigma_3,\beta_1,\sigma_1)=\frac{2\sin\pi b(\beta_1\mp(\sigma_1+\sigma_3-Q)-\frac{b}{2})\,\sin\pi b(\beta_1\mp(\sigma_3-\sigma_1)-\frac{b}{2})}{\sin\pi b(Q-2\beta_1)}\,,\tag{2.9}$$

$$c^M_{(\pm)}(\bar\sigma_3,e_1,\bar\sigma_1)=\frac{2\sin\pi b(e_1\mp(\bar\sigma_1+\bar\sigma_3-e_0)+\frac{b}{2})\,\sin\pi b(e_1\mp(\bar\sigma_3-\bar\sigma_1)+\frac{b}{2})}{\sin\pi b(e_0-2e_1)}\,.\tag{2.10}$$

and

$$c(\beta_2)=-\frac{\sin\pi b(Q-2\beta_2)}{\sin\pi b(2\beta_2)}\,.\tag{2.11}$$

The constants $\lambda_{_L}\,,\lambda_{_M}$ in (2.8) are the two bulk coupling constants, following the notation in [12]. Similarly one obtains the dual equation with $\tilde\lambda_{_L}=\lambda_{_L}^{1/b^2}\,,\tilde\lambda_{_M}=\lambda_{_M}^{-1/b^2}$

$$-\sqrt{\tilde\lambda_{_M}}\,\tilde c^M_{(\bar\delta)}(\bar\sigma_2,e_2,\bar\sigma_4)\,\hat{C}_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2-\frac{1}{2b} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}-\sqrt{\tilde\lambda_{_L}}\,\tilde c^L_{(\delta)}(\sigma_2,\beta_2,\sigma_4)\,\hat{C}_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2+\frac{1}{2b} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}$$

$$=\hat{C}_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1-\frac{1}{2b}\\ \sigma_4 & \sigma_1\end{bmatrix}+$$
$$\sqrt{\tilde\lambda_{_L}\,\tilde\lambda_{_M}}\,\tilde c(\beta_1)\,\tilde c^M_{(-\bar\delta)}(\bar\sigma_3=\bar\sigma_2-\frac{\bar\delta}{2b},e_1,\bar\sigma_1)\,\tilde c^L_{(-\delta)}(\sigma_3=\sigma_2-\frac{\delta}{2b},\beta_1,\sigma_1)\,\hat{C}_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{1}{2b}\\ \sigma_4 & \sigma_1\end{bmatrix},\tag{2.12}$$

replacing the constants in (2.9) (and (2.11)) with their duals, obtained by the change $b\to1/b$ (for β_i - fixed), while the dual of the matter constant (2.10) is obtained with $b\to-1/b$, so that

$$\tilde c^M_{(\mp)}(\bar\sigma_3,e,\bar\sigma_1)=\frac{2\sin\pi\frac{1}{b}(e-\frac{1}{2b}\mp(\bar\sigma_3+\bar\sigma_1-e_0))\,\sin\pi\frac{1}{b}(e-\frac{1}{2b}\mp(\bar\sigma_3-\bar\sigma_1))}{\sin\pi\frac{1}{b}(2e-e_0)}\,.\tag{2.13}$$

The two sets of equations (2.8),(2.12) are precisely the equations one obtains starting from a 4-point function with a ground ring generator added and then inserting the coefficients in the expansion of the product of the ground ring generator with the left or right tachyons (see formulae (A.36-A.38) of [12]; the computation there completes earlier partial results [15,16,17] for these OPE coefficients).

2.1. **Special case - trivial matter**

We choose as before the chiralities of type $(+-+)$. For trivial matter, i.e., a charge conservation condition with no screening charges,

$$e_0 = e_1 + (e_2 + \tfrac{b}{2}) + (e_0 - e_3) \equiv e_{12}^3 + e_0 + \tfrac{b}{2} \;\Rightarrow\; \beta_{23}^1 + \tfrac{b}{2} = Q \tag{2.14}$$

the matter boundary 3-point functions are trivial and the pure Liouville identity (2.3) $(t=+1)$ (normalised with the leg factors) with β_i restricted by (2.14) simplifies to

$$\begin{aligned}
&\hat{C}_{\sigma_3 \pm \frac{b}{2},\beta_3}\begin{bmatrix}\beta_2 - \frac{b}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix} = -\sqrt{\lambda_{_L}}\, c^L_{(\mp)}(\sigma_3,\beta_1,\sigma_1)\; \hat{C}_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}\\
&+ F^L_{++}\; \Gamma(\tfrac{1}{b}(Q-2\beta_2+b))\; \Gamma(b(Q-2\beta_1))\, \Gamma(b(2\beta_3-Q)\; C^L_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1-\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}\\
&= -\sqrt{\lambda_{_L}}\, c^L_{(\mp)}(\sigma_3,\beta_1,\sigma_1)\; \hat{C}^L_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{b}{2}\\ \sigma_4 & \sigma_1\end{bmatrix} + \frac{2\pi G_2(\sigma_3,\beta_2,\sigma_4)}{2\sin(\pi b(Q-2\beta_1))}\,.
\end{aligned} \tag{2.15}$$

We have used that for the values in (2.14) F^L_{++} has a zero $\sum_i \beta_i^{'} - Q \to 0$, while the Liouville reflected 3-point constant has a singularity with residue $1/2\pi$. Thus the second term in (2.15) reduces to the (leg-normalised) reflection Liouville amplitude [9],

$$\begin{aligned}
\Gamma(\tfrac{1}{b}(Q-2\beta))\,\Gamma(b(Q-2\beta))\, S(\sigma_2,\beta,\sigma_1) &= \frac{2\pi}{Q-2\beta}\, G_2(\sigma_2,\beta,\sigma_1)\,,\\
G_2(\sigma_2,\beta,\sigma_1) &= \frac{\lambda_L^{\frac{1}{2b}(Q-2\beta)}\; S_b(2\beta-Q)}{\prod_{s=\pm} S_b(\beta+s(\sigma_2+\sigma_1-Q))\, S_b(\beta+s(\sigma_2-\sigma_1))}\,,
\end{aligned} \tag{2.16}$$

$$G_2(\sigma_2,\beta,\sigma_1)\, G_2(\sigma_2,Q-\beta,\sigma_1) = S_b(2\beta-Q) S_b(Q-2\beta)\,,$$

where $S_b(x) = \Gamma_b(x)/\Gamma_b(Q-x)$ and $\Gamma_b(x)$ is the double Gamma function. The amplitude G_2 is the solution of the equations

$$\begin{aligned}
-\sqrt{\lambda_{_L}}\, c^L_{(\mp)}(\sigma_3,\beta_1,\sigma_1)\; G_2(\sigma_3,\beta_1+\tfrac{b}{2},\sigma_1) &= G_2(\sigma_3 \pm \tfrac{b}{2},\beta_1,\sigma_1)\,,\\
-\sqrt{\tilde{\lambda}_{_L}}\, \tilde{c}^L_{(\mp)}(\sigma_3,\beta_1,\sigma_1)\; G_2(\sigma_3,\beta_1+\tfrac{1}{2b},\sigma_1) &= G_2(\sigma_3 \pm \tfrac{1}{2b},\beta_1,\sigma_1)\,.
\end{aligned} \tag{2.17}$$

Since the general identity (2.8) should reduce for the values in (2.14) to the simpler identity (2.15), this implies a restriction on the unknown matter OPE coefficients involved in (2.8).

The identity (2.15) acquires a more symmetric form when rewritten for the cyclically symmetric correlator of type $(---)$ [2] obtained by two reflections (2.4), now written for

[2] This equation has been independently written down recently in [20].

the normalised correlators

$$\hat{C}^{\sigma_4,\sigma_2,\sigma_1}_{Q-\beta_3,\beta_2,\beta_1} = \frac{1}{b}\frac{1}{2\sin(\pi b(2\beta_3-Q))}\, G_2^{-1}(\sigma_1,\beta_3,\sigma_4)\, \hat{C}^{\sigma_4,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \\ = \frac{1}{b^2}\, \frac{\sin(\pi\frac{1}{b}(Q-2\beta_1))}{\sin(\pi b(Q-2\beta_3))}\, \frac{G_2(\sigma_2,\beta_1,\sigma_1)}{G_2(\sigma_1,\beta_3,\sigma_4)}\, \hat{C}^{\sigma_4,\sigma_2,\sigma_1}_{\beta_3,\beta_2,Q-\beta_1}\,. \tag{2.18}$$

We give for comparison the equations analogous to (2.17) for the nontrivial matter 2-point amplitude

$$-\sqrt{\lambda_{_M}}\, c^M_{(\mp)}(\bar\sigma_3,e,\bar\sigma_1)\ G_2^M(\bar\sigma_3,e-\tfrac{b}{2},\bar\sigma_1) = G_2^M(\bar\sigma_3\pm\tfrac{b}{2},e,\bar\sigma_1)\,, \\ -\sqrt{\tilde\lambda_{_M}}\ \tilde c^M_{(\mp)}(\bar\sigma_3,e,\bar\sigma_1)\, G_2^M(\bar\sigma_3,e+\tfrac{1}{2b},\bar\sigma_1) = G_2^M(\bar\sigma_3\pm\tfrac{1}{2b},e,\bar\sigma_1)\,. \tag{2.19}$$

3. Solutions

3.1. The matter factor

We shall start with the solution of the 2-point equations (2.19) for the matter degenerate values $2e = mb - n/b =: 2e_{m,n}$, where m,n are nonnegative integers, $m,n\in\mathbb{Z}_{\geq 0}$. The solution is expressed conveniently as

$$G_2^M(\bar\sigma_2,e,\bar\sigma_1) = \frac{(-1)^{(m+1)(n+1)}\ \lambda_{_M}^{\frac{2e-e_0}{2b}}}{S_b((m+2)b)\, S_b(\frac{n+2}{b})}\ \frac{G_M(\bar\sigma_1,e-b-(m+1)b,\bar\sigma_2)}{G_M(\bar\sigma_1,e-b+\frac{n+1}{b},\bar\sigma_2)} \\ = \lambda_{_M}^{\frac{2e-e_0}{2b}}\,\lambda_{_L}^{\frac{-(Q+mb+n/b}{2b)}}\, G_2(\bar\sigma_2+b,-e-\tfrac{n}{b},\bar\sigma_1+b)\frac{S_b(2Q+mb+\frac{n}{b})S_b(\frac{1}{b})}{S_b^2(\frac{n+2}{b})} \tag{3.1}$$

where

$$G_M(\bar\sigma_3,e_2,\bar\sigma_2) := S_b(-e_2+\bar\sigma_2+\bar\sigma_3)\, S_b(e_0-e_2+\bar\sigma_3-\bar\sigma_2)\,. \tag{3.2}$$

The representation of (3.1) in terms of $S_b(x)$ is not unique, but the expression is finite for the concrete values $e=e_{m,n}$ and reduces to a finite product of sin's. The equations (2.19) allow to extend the formula (3.1) to $m=n=-1$ and furthermore to the degenerate values $2e = e_0-(m+1)b+\frac{n+1}{b}$ with $m,n\in\mathbb{Z}_{\geq 0}$. [3]

• The solution of the pair of equations (2.8), (2.12) is given by a factorised expression combining the known Liouville expression [10] and a solution of the matter boundary pentagon equation. The solution of the matter boundary pentagon equation is a generalisation to generic b^2 of the solution in the rational $c<1$ case, where the fusing matrix is given [21] by a product of two basic ${}_4\Phi_3$ hypergeometric functions known to represent [22] the

[3] Note that unlike the Liouville case the analytic continuation of the two thermal cases $n=0$ or $m=0$ to generic values of e leads to different results, effectively inverse to each other.

quantum 6j symbols. The change of gauge affects only the prefactor. The non-rational generalisation is possible either if the representations are chosen to correspond to degenerate $c<1$ Virasoro representations, or, if a charge conservation condition with integer numbers of matter screening charges is imposed: to both we refer as "Coulomb gas" cases. The solutions in these cases are alternatively reproduced starting from the general formula of Ponsot and Teschner [10]. Thus to obtain the matter constant for

$$e_{123}-e_0\equiv e_1+e_2+e_3-e_0=mb-n/b\,,\quad m,n\in\mathbb{Z}_{\geq 0} \tag{3.3}$$

we start from the Liouville Coulomb gas expression for $\alpha_{123}-Q=-mb-n/b$ derived as a residue of the formula in [10]. We rewrite this particular solution of the Liouville pentagon equation (2.3) in terms of finite products of Gamma and sin functions and then continue analytically the result by replacing $b^2\to -b^2$, and $\alpha_i b\to e_i b$. The final result is a solution of the matter pentagon equation (2.5) and can be again expressed in compact form in terms of the ratios of double Gamma functions $\Gamma_b(x)$ using the notation (3.2),

$$\begin{aligned} &C^M_{\bar\sigma_2\,,e_0-e_3}\begin{bmatrix} e_2 & e_1\\ \bar\sigma_3 & \bar\sigma_1\end{bmatrix} = {}^M C^{\bar\sigma_3,\bar\sigma_2,\bar\sigma_1}_{e_3,e_2,e_1} = (-1)^{m+n}\,\lambda_M^{\frac{e_{123}-e_0}{2b}}\,\Pi_M(e_3,e_2,e_1)\times\\ &\frac{(-1)^{mn}S_b(b+2e_1-mb)}{S_b(b+2e_1+\frac{n}{b})}\sum_{k=0}^{m}\sum_{p=0}^{n}\frac{G_M(\bar\sigma_3,e_2-b-kb,\bar\sigma_2)\,G_M(\bar\sigma_3,e_0-e_3-b-\frac{n-p}{b},\bar\sigma_1)}{G_M(\bar\sigma_3,e_0-e_3-b+(m-k)b,\bar\sigma_1)G_M(\bar\sigma_3,e_2-b+\frac{p}{b},\bar\sigma_2)}\times\\ &\frac{S_b(b+2e_3-(m-k)b)\,S_b(b+2e_2-kb)\,S_b(\frac{1}{b}-2e_2-\frac{p}{b})\,S_b(\frac{1}{b}-2e_3-\frac{n-p}{b})}{S_b((k+1)b)\,S_b((m-k+1)b)\,S_b(\frac{p+1}{b})\,S_b(\frac{n-p+1}{b})}\,, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Pi_M(e_3,e_2,e_1) &= \frac{b^{Q(e_{123}-e_0)}\Gamma_b(b)S_b(\frac{n+1}{b})}{\Gamma_b(\frac{1}{b}+e_0-e_{123})}\prod_i\frac{\Gamma_b(\frac{1}{b}-2e_i)S_b(b+2e_i+\frac{n}{b})}{\Gamma_b(b+2e_i+e_0-e_{123})} =\\ &\frac{b^{Q(e_{123}-e_0)}\Gamma_b(\frac{1}{b})S_b(m+1)b)}{\Gamma_b(b-e_0+e_{123})}\prod_i\frac{\Gamma_b(b+2e_i)S_b(\frac{1}{b}-2e_i+mb)}{\Gamma_b(\frac{1}{b}-2e_i-e_0+e_{123})}\,. \end{aligned} \tag{3.5}$$

This formula is derived for generic values of $\{e_i\}$, subject of the constraint (3.3), but it reproduces as well the constants with degenerate values of e_i.

3.2. **The Liouville 3-point factor**

The matter charge conservation condition (3.3) rewrites as a relation for the Liouville labels, e.g., with the choice of chiralities $(+-+)$ one has

$$\beta_{13}^2\equiv\beta_1+\beta_3-\beta_2=(m+1)b-\tfrac{n}{b}\,. \tag{3.6}$$

In addition we choose also degenerate values for all matter labels, or equivalently, in terms of the Liouville labels β_i of the three fields in the correlator we take

$$\beta_i = b + m_i b - \tfrac{n_i}{b}\,, \;\; 2m_i, 2n_i \in \mathbb{Z}_{\geq 0}\,, \quad m, n \in \mathbb{Z}_{\geq 0}\,. \tag{3.7}$$

We further impose the (matter) fusion rule restriction that all $m_{ij}^k\,, n_{ij}^k\,, i \neq j \neq k \neq i$ are non-negative integers, so that $\sum_{i=1}^3 2m_i = 0$ mod 2. Other possible choices correspond to Liouville reflections $Q - \beta_i$ of some of the labels in (3.7) and the corresponding 3-point correlator is obtained with the help of the reflection relation (2.18).

For such values of $\{\beta_i\}$ the integral Ponsot - Teschner formula for the Liouville 3-point boundary constant simplifies. Taking into account two infinite series of poles it rewrites as a sum of two terms, each expressed in terms of a product of basic ${}_4\Phi_3$ hypergeometric functions, one given by a finite (of range n - as in (3.6)), the other - by an infinite, sum. A resummation of the infinite sums was performed in [18] in the particular case $m_i = 0\,, i = 1, 2, 3$ of (3.7).[4]

We shall follow here a different route to obtain a general simple formula without exploiting the integral PT representation. Namely we shall use recursively the Liouville equations (2.3) starting from the simplest correlator with three identical fields $\beta_i = b$

$$\begin{aligned} \hat{C}_{b,b,b}^{\sigma_3,\sigma_2,\sigma_1} &= \frac{2\pi\,\sqrt{\lambda_{_L}}^{\,-1}}{g_-(\sigma_1, b/2, \sigma_2)}\,\left(G_2(\sigma_3, b, \sigma_1) - G_2(\sigma_3, b, \sigma_2)\right) \\ &= \frac{2\pi\,\lambda_{_L}^{\frac{Q-3b}{2b}}}{S_b(\frac{2}{b})}\,\frac{\left(\tilde{c}_1(c_2 - c_3) + \tilde{c}_2(c_3 - c_1) + \tilde{c}_3(c_1 - c_2)\right)}{(c_2 - c_1)(c_1 - c_3)(c_3 - c_2)} \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &c_i = 2\cos\pi b(b - 2\sigma_i)\,, \quad \tilde{c}_i = 2\cos\pi\tfrac{1}{b}(\tfrac{1}{b} - 2\sigma_i)\,, \\ &g_-(\sigma_1, \beta, \sigma_2) = 2\cos\pi b(2\beta - 2\sigma_1) - 2\cos\pi b(b - 2\sigma_2) \end{aligned} \tag{3.9}$$

and recall that $\lambda_{_L}^{1/2}\cos\pi b(Q - 2\sigma)$ and $\tilde{\lambda}_{_L}^{1/2}\cos\pi(Q - 2\sigma)/b$ are the boundary cosmological constant and its dual.

The cyclic symmetry of the correlator is explicit in the second line of (3.8). This correlator, originally proposed in the microscopic approach of [11], and then reproduced in [18], is itself obtained directly from the (properly regularised) Liouville pentagon equation (2.3) for $t = 1, \beta_2 = b = Q - \beta_3, \beta_1 = b/2$, in a way similar to the derivation of the special case equation (2.15). For this choice of the parameters the l.h.s. and the first term in the r.h.s. of (2.3) are identified with reflected trivial Coulomb gas correlators, so that they are represented by 2-point amplitudes - the ones appearing in (3.8).

[4] The formal resummation in [18], which we believe is correct only when applied to the sum of the two terms, amounts to a relation for ${}_3\Phi_2$ q- Saalschutz type functions.

Let us first consider the "thermal" case with all $n_i = 0$ in (3.7). The Liouville correlator in (3.8) (normalised with the leg factors (2.1)) coincides with the tachyon correlator itself since it corresponds to a trivial matter condition with $m=0=n$ in (3.3), (3.6). Applying first trivial matter equations of the type in (2.15) we get the most general correlator with $m_{13}^2 = 0$. Then using the general equation (2.3) (for shifts of the pair (β_3, β_2)), we obtain, denoting $m = m_{13}^2 \,, s = m_{12}^3$,

$$
{}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} = \lambda_{_L}^{-1/2}\, \Pi_L(\beta_3,\beta_2,\beta_1)\, S_b(\tfrac{2}{b}) \times
$$

$$
\Big(\frac{G_2(\sigma_2+b,\beta_1,\sigma_1)}{g_-(\sigma_2,\beta_1-\frac{b}{2},\sigma_1)} \frac{S_b((s+1)b)}{S_b((s+m+1)b)} \sum_{p=0}^{s} \frac{S_b((p+m)b+Q)}{S_b(pb+Q)} \frac{G_2(\sigma_2+p\frac{b}{2},\beta_2-p\frac{b}{2},\sigma_3)}{G_2(\sigma_2+b+(p+m)\frac{b}{2},\beta_1-(p+m)\frac{b}{2},\sigma_1)}
$$

$$
+ \frac{G_2(\sigma_1-b,\beta_1,\sigma_2)}{g_-(\sigma_1,Q-\beta_1+\frac{b}{2},\sigma_2)} \frac{S_b((m+1)b)}{S_b((s+m+1)b)} \sum_{r=0}^{m} \frac{S_b((r+s)b+Q)}{S_b(rb+Q)} \frac{G_2(\sigma_1-r\frac{b}{2},\beta_3-r\frac{b}{2},\sigma_3)}{G_2(\sigma_1-b-(r+s)\frac{b}{2},\beta_1-(r+s)\frac{b}{2},\sigma_2)} \Big) \tag{3.10}
$$

where

$$
\Pi_L(\beta_3,\beta_2,\beta_1) = \frac{b^{e_0(Q-\beta_{123})}\Gamma_b(2Q-\beta_{123})\Gamma_b(Q-\beta_{23}^1)\Gamma_b(Q-\beta_{13}^2)\Gamma_b(Q-\beta_{12}^3)}{S_b(\frac{1}{b})S_b(\frac{2}{b})\,\Gamma_b(Q)\Gamma_b(Q-2\beta_1)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_3)}\,. \tag{3.11}
$$

In the overall prefactor in the product of the Liouville and matter correlators combining (3.5), (3.11) and the leg factor normalisation (2.1) the Γ_b functions are fully compensated, e.g. with the choice of the chiralities $(+,-,+)$ one has

$$
\begin{aligned}
&\Gamma(b(Q-2\beta_3))\Gamma(\tfrac{1}{b}(Q-2\beta_2))\Gamma(b(Q-2\beta_1))\,\Pi_M(\beta_3-b,-\beta_2+\tfrac{1}{b},\beta_1-b)\Pi_L(\beta_3,\beta_2,\beta_1)\\
&= \frac{2\pi\, S_b(2\beta_1-b)\, S_b(2\beta_2-\frac{1}{b})\, S_b(2\beta_3-b)}{S_b(2\beta_1-b-mb)\, S_b(2\beta_2-\frac{1}{b}-\frac{n}{b})\, S_b(2\beta_3-b-mb)} \frac{S_b(\frac{n+1}{b})}{S_b(\frac{1}{b})S_b(\frac{2}{b})}\,.
\end{aligned} \tag{3.12}
$$

In Appendix C we give a few explicit examples demonstrating the two formulae (3.4), (3.10). We shall rewrite now (3.10) in a form which reveals its symmetry under cyclic permutations. Let us first introduce some general notation

$$
\begin{aligned}
&G^{(-)}(\sigma_2,\beta,\sigma_1) := S_b(-\beta+\sigma_2+\sigma_1)S_b(Q-\beta+\sigma_2-\sigma_1) = (G^{(+)}(\sigma_2,Q-\beta,\sigma_1))^{-1}\,,\\
&\frac{G^{(\pm)}(\sigma_2,\beta-\frac{b}{2},\sigma_1)}{G^{(\pm)}(\sigma_2,\beta+\frac{b}{2},\sigma_1)} = g_\pm(\sigma_2,\beta,\sigma_1) = 2\sin\pi b(Q-2\beta)\ c_\pm^L(\sigma_2,\beta,\sigma_1)\,.
\end{aligned} \tag{3.13}
$$

For a non-negative integer k and an integer n of parity $p(n)$ denote

$$
B(\sigma_2,\sigma_1)^{(k;p(n))} := \frac{G^{(-)}(\sigma_2,-\frac{kb}{2}-\frac{n}{2b},\sigma_1)}{G^{(-)}(\sigma_2,b+\frac{kb}{2}-\frac{n}{2b},\sigma_1)} = (-1)^{(k+1)(n+1)}B(\sigma_1,\sigma_2)^{(k;p(n))} \tag{3.14}
$$

which is expressed as a $k+1$ order polynomial in $\{c_i\}$ using that for $k\neq 0$

$$g_-(\sigma_2, \tfrac{b}{2}-k\tfrac{b}{2}+\tfrac{n}{2b}, \sigma_1)\, g_-(\sigma_2, \tfrac{b}{2}+k\tfrac{b}{2}+\tfrac{n}{2b}, \sigma_1) = c_1^2+c_2^2-c_1\, c_2(-1)^n 2\cos\pi k b^2-(2\sin\pi k b^2)^2$$

while $B(\sigma_2,\sigma_1)^{(0;p(n))} = (-1)^n c_2 - c_3$. Similarly one defines the dual $\tilde{B}(\sigma_2,\sigma_3)^{(n;p(k))}$ so that the reflection amplitude is expressed as the ratio

$$\frac{\lambda_L^{\frac{2\beta_2-Q}{2b}} G_2(\sigma_2, \beta_2\!=\!b\!+\!m_2 b\!-\!n_2/b, \sigma_3)}{S_b(2\beta_2-Q)} = \frac{G^{(-)}(\sigma_2,\beta_2,\sigma_3)}{G^{(-)}(\sigma_2,Q-\beta_2,\sigma_3)} = \frac{\tilde{B}(\sigma_2,\sigma_3)^{(2n_2;p(2m_2))}}{B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}}\,. \tag{3.15}$$

Finally we introduce

$$\begin{aligned}
&P_2 \equiv P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} := \\
&(-1)^{m_{12}^3+2m_2}\,\lambda_L^{-\frac{m_{12}^3}{2}}\;\frac{S_b((2m_1+1)b)S_b((2m_2+1)b)}{S_b(b)}\sum_{p=0}^{m_{12}^3}\frac{S_b((m_{12}^3+1)b)}{S_b((p+1)b)\,S_b((m_{12}^3+1-p)b)}\times \\
&\qquad\frac{G_2(\sigma_2\!+\!p\frac{b}{2},\beta_2\!-\!p\frac{b}{2},\sigma_3)}{G_2(\sigma_2,\beta_2,\sigma_3)}\,\frac{G_2(\sigma_2\!-\!(m_{12}^3\!-\!p)\frac{b}{2},\beta_1\!-\!(m_{12}^3\!-\!p)\frac{b}{2},\sigma_1)}{G_2(\sigma_2,\beta_1,\sigma_1)} \\
&= \frac{(-1)^{2m_2}S_b((m_{12}^3+1)b)}{S_b(b)}\sum_{p=0}^{m_{12}^3}\frac{S_b(2m_1+(p-m_{12}^3+1)b)\,S_b(2m_2-(p-1)b)}{S_b((p+1)b\,S_b((m_{12}^3+1-p)b)}\times \\
&\qquad\frac{G^{(-)}(\sigma_2,\beta_2-pb,\sigma_3)}{G^{(-)}(\sigma_2,\beta_2,\sigma_3)}\,\frac{G^{(+)}(\sigma_2,\beta_1-(m_{12}^3-p)b,\sigma_1)}{G^{(+)}(\sigma_2,\beta_1,\sigma_1)}
\end{aligned} \tag{3.16}$$

and similarly P_1 and P_3, which can be obtained from (3.16) by cyclic permutations. The finite sum (3.16) is proportional to a truncated ${}_4\Phi_3$ type function. It can be expanded as a polynomial in the variables $\{c_i\}$.

With this notation (3.10) is cast in a form generalising the second line in (3.8),

$${}^L C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} = -\frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}}\,\Pi_L(\beta_3,\beta_2,\beta_1)}{B(\sigma_1,\sigma_2)^{(2m_1;0)}\,B(\sigma_2,\sigma_3)^{(2m_2;0)}\,B(\sigma_3,\sigma_1)^{(2m_3;0)}}\,F^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}\,,$$

$$\begin{aligned}
F^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &= (-1)^{2m_1}((-1)^{2m_2}\tilde{c}_2-\tilde{c}_3)B(\sigma_3,\sigma_1)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \\
&\qquad -(-1)^{2m_2}((-1)^{2m_3}\tilde{c}_3-\tilde{c}_1)B(\sigma_2,\sigma_3)^{(2m_2;0)}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}
\end{aligned} \tag{3.17}$$

$$= \tilde{c}_1 B(\sigma_3,\sigma_2)^{(2m_2;0)}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3} + \tilde{c}_2 B(\sigma_1,\sigma_3)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} + \tilde{c}_3 B(\sigma_2,\sigma_1)^{(2m_1;0)}P^{\sigma_1,\sigma_3,\sigma_2}_{\beta_1,\beta_3,\beta_2}\,.$$

In the second equality of (3.17) we have exploited the relation

$$B(\sigma_3,\sigma_1)^{(2m_3;0)}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} + \text{cyclic permutations} = 0 \tag{3.18}$$

which is proved using the alternative recursive derivations of (3.10), i.e., the cyclic symmetry, now explicit in (3.17.)

The composition of the reflection of all three fields with the reflection amplitude as in (2.4) and the duality transformation $b \to 1/b$ (changing notation $m_i \to n_i$) gives the correlator in the other thermal case when all $m_i = 0$ in (3.7). In that case the product of $B^{(0;p(2n_i))}$ replaces the denominator in (3.17) and the formula confirms the structure suggested in the microscopic approach of [11]. The dual polynomial $\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}$ is defined by changing in (3.16) $\beta_i \to Q-\beta_i\,, b\to 1/b\,, m_i \to n_i$. With the help of some identities for the basic hypergeometric functions one reproduces the formula found in [18] for the case $\{m_i = 0\,,\, n_i$ - integers$\}$ by exploiting in a formal way the PT formula. The expression in [18] is not explicitly symmetric under cyclic permutations, rather this symmetry is checked to hold on examples.

• To obtain the Liouville correlator defined for the general values (3.7) one can either use the dual pentagon equations, or, one can start from the correlator with all $m_i = 0$. In one of the steps the special case equation (2.15) has to be extended so that the second term in the r.h.s. is given by G_2 times a non-trivial Coulomb gas Liouville correlator. The final result is an expression generalising the first line in (3.17),

$$
\begin{aligned}
{}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} &= -\frac{\lambda_L^{\frac{Q-\beta_{123}}{2b}}\,\Pi'_L(\beta_3,\beta_2,\beta_1)}{B(\sigma_1,\sigma_2)^{(2m_1;p(n_1))}B(\sigma_2,\sigma_3)^{(2m_2;p(n_2))}B(\sigma_3,\sigma_1)^{(2m_3;p(n_3))}}\times \\
&(-1)^{2m_2 2n_1}\Big((-1)^{2m_1+2n_2}\tilde{B}(\sigma_2,\sigma_3)^{(2n_2;p(2m_2))}\tilde{P}^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}\; B(\sigma_3,\sigma_1)^{(2m_3;p(2n_3))}P^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1} \\
&-(-1)^{2m_2+2n_1}\tilde{B}(\sigma_3,\sigma_1)^{(2n_3;p(2m_3))}\tilde{P}^{\sigma_3,\sigma_2,\sigma_1}_{\beta_3,\beta_2,\beta_1}\; B(\sigma_2,\sigma_3)^{(2m_2;p(2n_2))}P^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3}\Big)\,,
\end{aligned}
\tag{3.19}
$$

$$
\Pi'_L(\beta_3,\beta_2,\beta_1) = \frac{(-1)^{m_{123}\,n_{123}+m_{123}+n_{123}}\,\Pi_L(\beta_3,\beta_2,\beta_1)\; S_b^3(\frac{1}{b})S_b(\frac{2}{b}-b)}{S_b(\frac{n^3_{12}+1}{b})S_b(\frac{n^1_{23}+1}{b})S_b(\frac{n^2_{13}+1}{b})S_b(\frac{n_{123}+2}{b}-b)}\,. \tag{3.20}
$$

Here, say, the polynomial P_2 is given by the first formula (3.16), where now all β_i are given by (3.7), with only the sign in front of (3.16) modified to $(-1)^{m^3_{12}(1+2n_3)+2m_3 2n_3+2m_2} = (-1)^{m_{123}2n_3+2m_1}$. Let us also write down the expression for one of the dual polynomials

$$
\begin{aligned}
\tilde{P}_1 \equiv \tilde{P}^{\sigma_2,\sigma_1,\sigma_3}_{\beta_2,\beta_1,\beta_3} &= \frac{(-1)^{n_{123}2m_2+2n_3}S_b(\frac{2n_1+1}{b})S_b(\frac{2n_3+1}{b})}{S_b(\frac{1}{b})}\sum_{u=0}^{n^2_{13}}\frac{S_b(\frac{n^2_{13}+1}{b})}{S_b(\frac{1+u}{b})S_b(\frac{n^2_{13}+1-u}{b})}\times \\
&\frac{G_2(\sigma_1+\frac{u}{2b},Q-\beta_1-\frac{u}{2b},\sigma_2)}{G_2(\sigma_1,Q-\beta_1,\sigma_2)}\,\frac{G_2(\sigma_1-\frac{n^2_{13}-u}{2b},Q-\beta_3-\frac{n^2_{13}-u}{2b},\sigma_3)}{G_2(\sigma_1,Q-\beta_3,\sigma_3)}\,.
\end{aligned}
\tag{3.21}
$$

Formula (3.19) gives the general expression for the Liouville factor in the tachyon 3-point boundary correlator with degenerate $c < 1$ representations. The cyclic symmetry of the full correlator is ensured by construction and is equivalent to a relation generalising (3.18),

$$(-1)^{2n_2(2m_2+1)} B(\sigma_3, \sigma_1)^{(2m_3;p(2n_3))}\, P_2 + \text{cyclic permutations} = 0 \tag{3.22}$$

and its dual with the dual polynomials and $m_i \leftrightarrow n_i$. In particular when all $m_i = 0$ the dual relation reproduces the cyclic identity satisfied by the first order dual polynomials $\tilde{B}(\sigma_2, \sigma_3)^{(0;p(2m_2))} = (-1)^{2m_2}\tilde{c}_2 - \tilde{c}_3$, etc., which appear in the numerator in (3.17). The composition of duality transformation $b \to 1/b, m_i \leftrightarrow n_i$ with reflection of all three fields keeps (3.19) invariant.

- We conclude with some remarks.

The above solutions of the Liouville and matter equations defined for generic values of the parameters apply in particular to the rational (minimal gravity) theory in which case there may appear further truncations of the sums.

The (thermal) matter 3-point function (3.4) is given by the same basic hypergeometric function as one of the polynomials P_i in the numerator of the Liouville factor with proper identification of the parameters

$${}^{M}C^{\bar{\sigma}_3,\bar{\sigma}_2,\bar{\sigma}_1}_{e_3,e_2,e_1} \sim P^{\frac{1}{b}-\bar{\sigma}_1,\frac{1}{b}-\bar{\sigma}_3,\frac{1}{b}-\bar{\sigma}_2}_{\frac{1}{b}-e_1,\, e_3+b,\, e_2+b} = P_3\,. \tag{3.23}$$

Similarly for $e_1 = m_1 b\,, e_3 = m_3 b\,, e_2 = e_0 - m_2 b$ (3.4) is identified with the polynomial P_1 in (3.17,) etc. Analogous to (3.23) formulae hold for the case $\beta_i = b - n_i/b$, relating (3.4) to one of the dual polynomials with $\sigma_i = \bar{\sigma}_i + b$.

The factorised matter - Liouville correlator contains "too many" boundaries - their cardinality should be the same as that of the set of tachyons. Examples of a linearly independent set of boundaries is provided by the "trivial matter boundaries", i.e., one σ_i is set to zero, while the intermediate two are fixed by the fusion rules: the matter factor is reduced to a correlator of chiral vertex operators. On the level of 1-point functions or boundary states one can represent the states with general degenerate matter boundaries $\bar{\sigma}_i$ as linear combinations of FZZ states with shifted boundary parameters σ_i [23]. It remains to look for some lifting of this fusion type relation to the boundary correlators, see the recent work [24] for a step in this direction.

Another choice, the consistency of which deserves to be investigated, are the "tachyonic boundaries" when the pairs $(\bar{\sigma}_i, \sigma_i)$ themselves satisfy the mass-shell condition required by BRST invariance. Such correlators could be rather interpreted as the "string q-6j symbols", i.e., the OPE coefficients of the string CVO. They satisfy the pair of equations (2.8),(2.12), with correlated signs $\delta, \bar{\delta}$, preserving the chosen (chirality) type of the

relation. For such a "tachyonic" choice the matter boundary parameters are to be extended to the complex values $2\bar{\sigma}_i = e_0 \pm iP$ corresponding to the FZZ branes $2\sigma_i = Q - iP$.

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Appendix A. Data on fundamental OPE coefficients

The fusing matrix elements and the boundary OPE constants in (2.3), (2.5) containing a fundamental $c > 25$ or $c < 1$ Virasoro representation are known Coulomb gas constants, e.g.,

$$\begin{aligned} F^L_{s,t} &= F^L_{\beta_1 - s\frac{b}{2}, \beta_2 - t\frac{b}{2}} \begin{bmatrix} \beta_2 & -\frac{b}{2} \\ \beta_3 & \beta_1 \end{bmatrix} \\ &= \frac{\Gamma(tb(Q-2\beta_2))\,\Gamma(1-sb(Q-2\beta_1))}{\Gamma(\frac{1-s}{2}+tb(\beta_3-\beta_2+st\beta_1-st\frac{b}{2}))\,\Gamma(\frac{t+1}{2}-tb(\beta_2+\beta_3-st\beta_1-\frac{b}{2})-\frac{s-t}{2}bQ)}\,. \end{aligned} \tag{A.1}$$

The dual fusion matrix elements $\tilde{F}^L_{s,t}$ are obtained with $b \to 1/b$. All these expressions should be considered as furthermore restricted by the fusion rules. The gauge choice is such that e.g., $F^L_{++} = 1$ if $\beta_1 = 0$ since the fusion rule leads to $\beta_3 = \beta_2 - b/2$, or, if, $\beta_3 = Q$ so that $\beta_1 - b/2 = Q - \beta_2$. The expression for the matter fundamental fusion matrix elements is obtained from (A.1) by analytic continuation $b^2 \to -b^2$ and $b\beta_i \to be_i$ (so that $b(\beta_1 - tb/2) \to b(e + tb/2)$)

$$\begin{aligned} F^M_{s,t} :=& F^M_{e_1 - s\frac{b}{2}, e_2 - t\frac{b}{2}} \begin{bmatrix} e_2 & \frac{b}{2} \\ e_3 & e_1 \end{bmatrix} \\ &= \frac{\Gamma(tb(2e_2-e_0))\,\Gamma(1+sb(e_0-2e_1))}{\Gamma(\frac{1+s}{2}-tb(e_3-e_2+ste_1+st\frac{b}{2}))\,\Gamma(\frac{1-t}{2}+tb(e_2+e_3-ste_1+\frac{b}{2})+\frac{s-t}{2}be_0)}\,. \end{aligned} \tag{A.2}$$

The dual $\tilde{F}^M_{s,t}$ is recovered from $F^M_{-s,-t}$ by the change $b \to -1/b$. For the choice of the chiralities of the three fields as in (2.6) one has $F^M_{s,t} = F^L_{-s,t}\,,\ \ \tilde{F}^M_{s,t} = \tilde{F}^L_{-s,t}$ which implies (2.7). Furthermore one needs the particular fundamental constants in (2.3) and (2.5). In the Liouville case the constant is given by [9]

$$C^L_{\sigma_3 \pm \frac{b}{2}\,\beta_1 + \frac{b}{2}} \begin{bmatrix} -\frac{b}{2} & \beta_1 \\ \sigma_3 & \sigma_1 \end{bmatrix} = -\frac{b^2\sqrt{\lambda_L}\,\Gamma(1-2b\beta_1)}{\Gamma(1+(Q-2\beta_1)b)}\, c^L_{(\mp)}(\sigma_3, \beta_1, \sigma_1) \tag{A.3}$$

with the last constant written down in (2.9). The corresponding matter factor (obtained also as analytic continuation of (2.9)) reads

$$C^M_{\bar\sigma_3\mp\frac{b}{2},e_1-\frac{b}{2}}\begin{bmatrix}\frac{b}{2} & e_1\\ \bar\sigma_3 & \bar\sigma_1\end{bmatrix}=b^2\,\frac{\sqrt{\lambda_{_M}}\,\Gamma(1-2be_1)}{\Gamma(1+(e_0-2e_1)b)}\,c^M_{(\pm)}(\bar\sigma_3,e_1,\bar\sigma_1) \tag{A.4}$$

with the explicit expression given in (2.10). Combining (A.3), (2.9) and (A.4), (2.10) the product of the coefficients in the l.h.s. of the $t=t'=-1$ identities in (2.3), (2.5) reads

$$\begin{aligned}&C^M_{\bar\sigma_3\,e_2-\frac{b}{2}}\begin{bmatrix}e_2 & \frac{b}{2}\\ \bar\sigma_4 & \bar\sigma_2\end{bmatrix}\;C^L_{\sigma_3\,\beta_2+\frac{b}{2}}\begin{bmatrix}\beta_2 & -\frac{b}{2}\\ \sigma_4 & \sigma_2\end{bmatrix}\\
&=C^M_{\bar\sigma_3=\bar\sigma_2+\bar\delta\frac{b}{2},e_2-\frac{b}{2}}\begin{bmatrix}\frac{b}{2} & e_2\\ \bar\sigma_2 & \bar\sigma_4\end{bmatrix}\;C^L_{\sigma_3=\sigma_2\,-\delta\frac{b}{2}\,\beta_2+\frac{b}{2}}\begin{bmatrix}-\frac{b}{2} & \beta_2\\ \sigma_2 & \sigma_4\end{bmatrix}\\
&=-\sqrt{\lambda_{_L}\,\lambda_{_M}}\,\frac{\Gamma(\frac{1}{b}(Q-2\beta_2-b))}{\Gamma(\frac{1}{b}(Q-2\beta_2+b))}\;c(\beta_2)\,c^M_{(-\bar\delta)}(\bar\sigma_2,e_2,\bar\sigma_4)\,c^L_{(\delta)}(\sigma_2,\beta_2,\sigma_4)\,.\end{aligned} \tag{A.5}$$

The Gamma's in (A.5) are eliminated by the leg factor normalisation (2.1) and collecting everything we obtain the relation (2.8) for the normalised constants $\hat C$

$$\frac{1}{\Gamma(b(2\beta_3-Q))}\hat C_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}=\Gamma(\tfrac{1}{b}(Q-2\beta_2))\Gamma(b(Q-2\beta_1))C^L_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}C^M_{\bar\sigma_3\,,e_3}\begin{bmatrix}e_2 & e_1\\ \bar\sigma_4 & \bar\sigma_1\end{bmatrix}$$

Appendix B. The equation in the nonstandard Liouville gravity

In [12] another version of the Liouville gravity has been constructed trading the standard matter screening charges for non-trivial tachyon interaction terms, with "diagonal" matter screening charges $e_0=1/b-b$. It had led to functional equations with shifts along the diagonal $e\pm e_0$ and the solutions for the 4-point tachyon bulk correlators in this model were confirmed by a matrix model based construction [19]. To obtain an equation for the boundary 3-point correlator with the OPE projected to the diagonal shifts $e\pm e_0$ we cannot follow the derivation of (2.8),(2.12) above by linear combinations of matter and Liouville pentagon equations. Instead we shall exploit the ground ring relations of [12], composing the individual terms in these relations. E.g., taking the order ${}^{\sigma_4}B^{\sigma_3}_{\beta_2}a_-^{\sigma_2'=\sigma_3'}a_+^{\sigma_2}B^{\sigma_1}_{\beta_1}$ in the product of the tachyons with the ground ring generators $a_\pm$ one obtains (cancelling an

overall sign)

$$\sqrt{\tilde\lambda_{_M}\lambda_{_M}\lambda_{_L}}\,\tilde c^M_{(\bar\delta)}(\bar\sigma_2{=}\bar\sigma_3-\tfrac{\bar\delta' b}{2}+\tfrac{\bar\delta}{2b},e_2-\tfrac{b}{2},\bar\sigma_4)\,c(\beta_2)c^M_{(-\bar\delta')}(\bar\sigma_3-\frac{\bar\delta' b}{2},e_2,\bar\sigma_4)c^L_{(\delta')}(\sigma_3+\frac{\delta' b}{2},\beta_2,\sigma_4)\times$$
$$\hat C_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2-\frac{e_0}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}+\sqrt{\tilde\lambda_{_L}}\,\tilde c^L_{(\delta)}(\sigma_2=\sigma_3+\tfrac{\delta}{2b}+\tfrac{\delta' b}{2},\beta_2-\tfrac{b}{2},\sigma_4)\,\hat C_{\sigma_2\,,\beta_3}\begin{bmatrix}\beta_2+\frac{e_0}{2} & \beta_1\\ \sigma_4 & \sigma_1\end{bmatrix}$$

$$=\sqrt{\lambda_{_M}\tilde\lambda_{_L}\tilde\lambda_{_M}}\,c^M_{(\bar\delta')}(\bar\sigma_3,e_1+\tfrac{1}{2b},\bar\sigma_1)\,\tilde c(\beta_1)\,\tilde c^M_{(-\bar\delta)}(\bar\sigma_2-\frac{\bar\delta}{2b},e_1,\bar\sigma_1)\,\tilde c^L_{(-\delta)}(\sigma_2-\frac{\delta}{2b},\beta_1,\sigma_1)\ \times$$
$$\hat C_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1+\frac{e_0}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}+\sqrt{\lambda_{_L}}\,c^L_{(-\delta')}(\sigma_3,\beta_1-\tfrac{1}{2b},\sigma_1)\,\hat C_{\sigma_3\,,\beta_3}\begin{bmatrix}\beta_2 & \beta_1-\frac{e_0}{2}\\ \sigma_4 & \sigma_1\end{bmatrix}\ . \tag{B.1}$$

The opposite order leads to a minus sign for each term so that the final relation does not change. We may restrict to diagonal shifts of the boundary labels as well, taking $\delta'=-\delta\,,\ \bar\delta'=\bar\delta$.

Appendix C. Examples

Example 1: $e_{123}=e_0+b=1/b$

The matter formula (3.4) reads (set $\lambda_{_M}=1$)

$$C^M_{\bar\sigma_2\,,e_0-e_3}\begin{bmatrix}e_2 & e_1\\ \bar\sigma_3 & \bar\sigma_1\end{bmatrix}=-\frac{\Pi_M(e_3,e_2,e_1)}{2\sin\pi b^2 2\sin\pi 2e_1 b}\left(c^M_{(-)}(\bar\sigma_3,e_3-b/2,\bar\sigma_1)+c^M_{(+)}(\bar\sigma_3,e_2-b/2,\bar\sigma_2)\right)$$
$$=\frac{b^2\Gamma(b^2)\,\prod_i\Gamma(1-2e_i b)}{(2\pi)^2}2(\sin\pi 2e_2 b\,c^M_1+\sin\pi 2e_3 b\,c^M_2+\sin\pi 2e_1 b\,c^M_3) \tag{C.1}$$

where $c^M_i=2\cos\pi b(b+2\bar\sigma_i)$. The cyclic symmetry of the 3-point function is explicit. As a particular example one recovers from (3.4) the OPE constant (A.4), leading to (2.10).

We shall use (C.1) for the particular choice of three degenerate matter fields

$$\beta_1=\beta_2=\beta_3=2b\to e_1=e_3=b=e_0-e_2 \tag{C.2}$$

or any other choice of two $(+)$ and one $(-)$ chiralities. For our example $k=2$ in (3.14) is even and the polynomial (3.14) in the variables c_2,c_1

$$B(\sigma_2,\sigma_1)^{(2;0)}=(c_2-c_1)g_-(\sigma_2,-\tfrac{b}{2},\sigma_1)g_-(\sigma_2,\tfrac{3b}{2},\sigma_1)=(c_2-c_1)P(c_2,c_1) \tag{C.3}$$

is antisymmetric. The polynomial (3.16) is symmetric in σ_1,σ_3 and is proportional to

$$P_2:=g_-(\sigma_2,-\tfrac{b}{2},\sigma_1)+g_-(\sigma_2,\tfrac{3b}{2},\sigma_3)=-\sum_{i=1}^3 c_i+c_2\,(1+2\cos\pi 2b^2)\,. \tag{C.4}$$

Then the Liouville factor (3.17) reads

$$
{}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{2b,2b,2b} =
$$
$$
= \frac{S_b(3b)S_b(2b)}{S_b^2(b)} \frac{\lambda_L^{\frac{Q-6b}{2b}}\ \Pi_L(2b,2b,2b)}{B(\sigma_1,\sigma_2)^{(2;0)}\, B(\sigma_2,\sigma_3)^{(2;0)}\, B(\sigma_3,\sigma_1)^{(2;0)}} \det \begin{pmatrix} \tilde{c}_3\, X_3 & \tilde{c}_2\, X_2 & \tilde{c}_1\, X_1 \\ c_3 & c_2 & c_1 \\ 1 & 1 & 1 \end{pmatrix} \tag{C.5}
$$

with $X_3 = X_3(c_1,c_2,c_3) := P_3\, P(c_1,c_2)$. Combining (C.5) with (C.1) and the full prefactor from (3.12) one obtains the tachyon correlator in this example. Note that for the choice of the chiralities $-\epsilon_1 = 1 = \epsilon_2 = \epsilon_3$, the matter correlator (C.1) is indeed proportional to the polynomial P_3 in (C.4), since all c_k^M are identified with c_k. Similarly the choice of the negative chirality as $\epsilon_2 = -1$ or $\epsilon_3 = -1$ leads to the polynomial P_1 or P_2 respectively.

Example 2: $e_{123} = e_0 - \frac{1}{b} = -b$

The matter formula (3.4) reads

$$
C^{M}_{\bar{\sigma}_2\,,e_0-e_3}\begin{bmatrix} e_2 & e_1 \\ \bar{\sigma}_3 & \bar{\sigma}_1 \end{bmatrix} = \frac{\Pi_M(e_3,e_2,e_1)}{2\sin\pi/b^2 2\sin\pi 2e_1/b}\Big(\tilde{c}^M_{(+)}(\bar{\sigma}_3, e_3+\tfrac{1}{2b},\bar{\sigma}_1) + \tilde{c}^M_{(-)}(\bar{\sigma}_3, e_2+\tfrac{1}{2b},\bar{\sigma}_2)\Big)
$$
$$
= \frac{1}{b^2}\frac{\Gamma(\frac{1}{b^2})\prod_i\Gamma(1+\frac{2e_i}{b})}{(2\pi)^2} 2(\sin\pi(-\tfrac{2e_2}{b})\,\tilde{c}_1^M + \sin\pi(-\tfrac{2e_3}{b})\,\tilde{c}_2^M + \sin\pi(-\tfrac{2e_1}{b})\,\tilde{c}_3^M)\,, \tag{C.6}
$$

where $\tilde{c}_i^M = 2\cos\pi\frac{1}{b}(\frac{1}{b} - 2\bar{\sigma}_i)$. Comparing with (C.1) one observes that the symmetry $b \to -1/b$ of the correlator is indeed confirmed. The matter correlator (C.6) can be used e.g., to compute the tachyon 3-point function with

$$
\beta_3 = \beta_2 = \beta_1 = b - 1/b \Rightarrow e_1 = e_3 = -1/b = e_0 - e_2\,.
$$

The Liouville 3-point function in this case has been given in [18] and it is cast in a form similar to (C.5),

$$
{}^{L}C^{\sigma_3,\sigma_2,\sigma_1}_{b-\frac{1}{b},b-\frac{1}{b},b-\frac{1}{b}} = \frac{S_b(\frac{1}{b})S_b(\frac{3}{b})}{S_b(\frac{2}{b})S_b(\frac{5}{b})} \frac{\lambda_L^{\frac{Q-3e_0}{2b}}\ \Pi_L(b-\frac{1}{b},b-\frac{1}{b},b-\frac{1}{b})}{(c_1-c_2)(c_2-c_3)(c_3-c_1)} \det \begin{pmatrix} c_3\tilde{X}_3 & c_2\tilde{X}_2 & c_1\tilde{X}_1 \\ \tilde{c}_3 & \tilde{c}_2 & \tilde{c}_1 \\ 1 & 1 & 1 \end{pmatrix} \tag{C.7}
$$

where $\tilde{X}_i$ is the dual ($b \to 1/b$) of the polynomial X_i in (C.5). The duality $b \to 1/b$ transformation of (C.7), so that $\beta_i = b - 1/b \to 1/b - b$, gives a new correlator, which is obtained alternatively from (C.5) by reflecting all three boundary fields $\beta_i = 2b \to Q-2b = 1/b - b$ with the corresponding 2-point reflection amplitudes.

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